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RESONANCE MOTIONS IN AN ESSENTIALLY NONLINEAR SYSTEM CONTAINING STABLE ELEMENTS

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A perturbed rotary-oscillatory system with many degrees of freedom, subjected to periodic external forces is investigated. By successive approximations with respect to a small parameter a resonance solution on the infinite time interval is constructed and its Liapunov stability is considered. Calculations for an actual example of a system with three degrees of freedom simulating the influence of a spinning unbalanced rotor on the foundation, are carried out.

1. Statement of the problem. We examine a nonlinear system leading to equations with rotating phase of the form

$$
\begin{aligned}
a^{\cdot} &= f(t, a, \psi, h, \varepsilon) \\
\psi^{\cdot} &= \omega(a) + F(t, a, \psi, h, \varepsilon) \\
h^{\cdot} &= H(a)\, h + g(t, a, \psi, h, \varepsilon)
\end{aligned}
\tag{1.1}
$$

Here $t \in [t_0, \infty)$ is the independent variable (time), $a \in G_a$ is quasi-static variable (energy or amplitude), $\psi \in [\psi_0, \infty)$ is the phase of the oscillations or of the rotations, $h = (h_1, \ldots, h_r)$ is a stable vector, $h \in G_h$, where G_h is some neighbourhood of the origin, $\varepsilon > 0$ is a small parameter, $\varepsilon \in [0, \varepsilon_0]$. It is assumed that the right-hand side of system (1.1) satisfied the conditions:

1) it is defined and real in the region indicated;

2) the functions f, F and g are periodic in t and ψ with periods τ and 2π respectively,

3) the frequency $\omega(a)$ is nonnegative, and for certain relatively prime numbers m and n there exists the solution a^* of the equation $m\omega = n\nu$ $(m \geqslant 1)$, where $\nu = 2\pi / \tau$ is the frequency of the external forces;

4) $H(a^*)$ is a stable matrix, i.e., all r roots of the equation $\Delta(\lambda) = \det(H(a^*) - I\lambda) = 0$, where I is the unit matrix, satisfy the condition $\operatorname{Re} \lambda_k(a^*) \leqslant \lambda_0 < 0$ $(k = 1, \ldots, r)$;

5) it is continuous in t; in the region being examined the functions f, ω, F and g have partial derivatives upto second order with respect to the rest of the arguments, while matrix H has a first-order partial derivative, all of which satisfy Lipschitz conditions with constants not depending on t

6) the estimates

$$f, F = O(|h| + |\varepsilon|), \quad g = O(|h|^2 + |\varepsilon|) \tag{1.2}$$

are valid.

The nonlinear system $x^{\cdot} = X(x) + \varepsilon f(t, x, \varepsilon)$, x is a vector, reduces to a system of form (1.1) with estimates of a more special form for the right-hand side: $f, F, g = O(|h|^2 + |\varepsilon|)$, x is a vector under the assumption that the generating system admits of a stable two-parameter family of rotary-oscillatory motions [1-3]. The author investigated such a system in [2]. Systems of type (1.1) were investigated in [1] by the averaging method and in [3] by the method of local integral manifolds. Here we pose and solve the problem of constructing individual resonance solutions of system (1.1) with estimates (1.2) on an unbounded time interval by the constructive scheme of successive approximations in the small parameter method [4] and we investigate the Liapunov stability of these solutions.

2. Construction of a resonance solution. A steady-state resonance solution of system (1.1) of form m / n is constructed as the sum of a generating motion and a perturbing motion [2]

$$\begin{aligned} &a = a^* + \varepsilon x(t, \varepsilon), \quad \psi = (n / m)\nu(t - t_0) + \varphi + \varepsilon y(t, \varepsilon) \\ &h = \varepsilon z(t, \varepsilon) \end{aligned} \tag{2.1}$$

Here, x, y, z are unknown periodic functions of t of period $T = m\tau$, φ is a parameter (the phase constant). The substitution of (2.1) into (1.1) leads to a quasi-linear system in x, y, z, whose periodic solution is constructed by successive approximations of powers of ε [2, 4]

$$\begin{aligned}
&\dot{x}_{i+1} = (f'_\varepsilon) + (f_h')z_{i+1} + \varepsilon[(f''_{a\varepsilon})x_i + (f''_{\psi\varepsilon})y_i + (f''_{h\varepsilon})z_i + \\
&+ {}^1/_2(f_{\varepsilon^2}'') + (f''_{ah})x_i z_i + (f''_{\psi h})y_i z_i + {}^1/_2(f_{h^2}'')z_i^2 + f^*(t, x_i, y_i, z_i, \varepsilon)] \\
&\dot{y}_{i+1} = \omega'(a^*)x_{i+1} + (F_\varepsilon') + (F_h')z_{i+1} + \varepsilon[{}^1/_2\omega''(a^*)x_i^2 + \\
&+ (F''_{a\varepsilon})x_i + (F''_{\psi\varepsilon})y_i + (F''_{h\varepsilon})z_i + {}^1/_2(F_{\varepsilon^2}'') + (F''_{ah})x_i z_i + \\
&+ (F''_{\psi h})y_i z_i + {}^1/_2(F_{h^2}'')z_i^2 + F^*(t, x_i, y_i, z_i, \varepsilon)] \\
&\dot{z}_{i+1} = H(a^*)z_{i+1} + (g_\varepsilon') + \varepsilon[(g''_{a\varepsilon})x_i + (g''_{\psi\varepsilon})y_i + (g''_{h\varepsilon})z_i + \\
&+ {}^1/_2(g_{\varepsilon^2}'') + {}^1/_2(g_{h^2}'')z_i^2 + H'(a^*)x_i z_i + g^*(t, x_i, y_i, z_i, \varepsilon)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 0, 1, 2 \ldots)
\end{aligned} \tag{2.2}$$

Here an expression of the type (f_ε') signifies that the derivatives are computed for $\varepsilon = 0$ and for the generating solution: $a = a^*$, $\psi = (n / m)\nu(t - t_0) + \psi_0$, $h = 0$; f^*, F^*, g^* are known functions satisfying Lipschitz conditions in the variables x, y, z, ε with constants independent of t and vanishing identically at $\varepsilon = 0$.

It should be noted that the mentioned quasi-linear system of equations for x, y, z has the form of system (2.2) in which the indices $i + 1$ and i have been dropped. In fact, after the substitution of expressions (2.1) into (1.1), the expansion with respect to ε, and the division by $\varepsilon \neq 0$, equations of form (2.2) follow as a consequence of the assumptions made in (5) of Sect. 1 concerning the smoothness of the right-hand sides and on the basis of estimates (1.2). When the functions f and F have a quadratic estimate with respect to h [2] as the function g does (see (1.2)), the form of system (2.2) is simpler since the derivatives of type (f_h'), (f_{ah}''), $(f_{\psi h}'')$ and the analogous ones for function F identically equal zero. Integration at each step of the variables x_{i+1} and y_{i+1} is carried out explicitly and independently of the vector h_{i+1}. The investigation of the stability of the resonance solution (2.1) simplifies considerably (in comparison the one in Sect. 3 below), since it turns out that the variable h has no influence on the stability in the first approximation being examined [2].

As the zero approximation we take the periodic solution of system (2.2) when $\varepsilon = 0$

$$z_0 = z_0^*(t, \varphi) = H[(g_\varepsilon')], \quad H[f] \equiv \int_{-\infty}^{t} \exp H(a^*)(t - t_1) f dt_1$$

$$x_0 = A_0 + \int_{t_0}^{t} [(f_\varepsilon') + (f_h') z_0] dt_1 \equiv A_0 + x_0^*(t, \varphi) \tag{2.3}$$

$$y_0 = B_0 + \omega'(a^*) A_0 (t - t_0) + \int_{t_0}^{t} [\omega'(a^*) x_0^* + (F_\varepsilon') + (F_h') z_0] dt_1 \equiv B_0 + y_0^*(t, \varphi)$$

Here φ, A_0 and B_0 are parameters selected so that the resulting solution is periodic. From the periodicity condition on x_0^* follows an equation in the parameter φ

$$P(\varphi) \equiv \int_0^T [(f_\varepsilon') + (f_h') H[(g_\varepsilon')]] dt = 0 \tag{2.4}$$

If $\varphi^* (\mathrm{mod}\ 2\pi / m)$ is a real root of the transcendental Eq. (2.4), then the function x_0 is periodic for any A_0, including

$$A_0 = -[\omega'(a^*) T]^{-1} \int_0^T \omega'(a^*) x_0^* + (F_\varepsilon') + (F_h') z_0] dt \quad (\omega'(a^*) \neq 0)$$

For the A_0 indicated the function y_0 is periodic for any B_0. As a result the periodic functions x_0 and z_0 are completely determined, while the parameter B_0 is found from the conditions of the periodicity of the first approximation resulting from (2.2) when $f^* \equiv 0$, $F^* \equiv 0$, $g^* \equiv 0$

$$z_1 = z_0 + \varepsilon z_1^{**} + \varepsilon B_0 H[(g_{\psi\varepsilon}'')] \equiv z_0 + \varepsilon z_1^* \tag{2.5}$$

$$x_1 = x_0 + \varepsilon A_1 + \varepsilon \int_{t_0}^{t} f_1(t_1) dt_1 + \varepsilon B_0 \int_{t_0}^{t} [(f_{\psi\varepsilon}'') + (f_{\psi h}'') z_0 + (f_h') H[(g_{\psi\varepsilon}'')] dt_1 \equiv x_0 + \varepsilon A_1 + \varepsilon x_1^*$$

$$y_1 = y_0^* + B_1 + \varepsilon y_1^*$$
$$z_1^{**} = H\,[(g_{a\varepsilon}'')\,x_0 + (g_{\psi\varepsilon}'')\,y_0^* + (g_{h\varepsilon}'')\,z_0 + {}^1/_2\,(g_{\varepsilon^2}'') + {}^1/_2\,(g_{h^2}'')z_0^2 + H'(a^*)\,x_0 z_0]$$
$$f_1 = (f_h')\,z_1^{**} + (f_{a\varepsilon}'')\,x_0 + (f_{\psi\varepsilon}'')\,y_0^* + (f_{h\varepsilon}'')\,z_0 + {}^1/_2\,(f_{\varepsilon^2}'') + (f_{ah}'')\,x_0 z_0 + (f_{\psi h}'')\,y_0^* z_0 + {}^1/_2\,(f_{h^2}'')\,z_0^2$$

The parameter B_0 is determined from the periodicity condition on x_1 under the condition that φ^* is a simple root of (2.4)

$$B_0 = -\frac{1}{P'(\varphi^*)}\int_0^T f_1(t)\,dt \tag{2.6}$$

The constant A_1 is found from an analogous relation for y_1^*

$$y_1^* = \omega'(a^*)\,A_1(t - t_0) + \int_{t_0}^{t}\Big[\omega'(a^*)\,x_1^* + \frac{1}{2}\,\omega''(a^*)\,x_0^2 + (F_h')z_1^* + (F_{a\varepsilon}'')\,x_0 + (F_{\psi\varepsilon}'')\,y_0 + (F_{h\varepsilon}'')\,z_0 + \frac{1}{2}\,(F_{\varepsilon^2}'') + (F_{ah}'')x_0 z_0 + (F_{\psi h}'')\,y_0 z_0 + \frac{1}{2}\,(F_{h^2}'')\,z_0^2\Big]\,dt_1 \equiv \omega'(a^*)\,A_1(t - t_0) + y_1^{**}(t)$$
$$A_1 = -\,y_1^{**}(T)\,/\,\omega'(a^*)\,T$$

Here the functions x_1^* and z_1^* are completely determined after the substitution of B_0 into (2.5). As a result we have found the following approximations: y_0, x_1 and z_1, while the expression for y_1 is of form (2.5) wherein $B_1(0) = B_0$. The constant $B_1(\varepsilon)$ must be arranged such that the succeeding approximation is periodic. And so on.

The proposed scheme (2.2) enables us to find periodic functions with arbitrarily high index i, i.e., any formal approximation in powers of ε of the periodic functions (2.1). In fact, suppose that this statement is valid for $i = 0, 1, \ldots, k - 1$, i.e.,

$$x_{k-1} = x_0 + \varepsilon A_{k-1} + \varepsilon x_{k-1}^*, \quad y_{k-1} = y_0^* + B_{k-1} + \varepsilon y_{k-1}^*$$
$$z_{k-1} = z_0 + \varepsilon z_{k-1}^*$$

is a solution of the system of the $(k-1)$ st approximation, where the constant $B_{k-1}(\varepsilon)$ has been found from the periodicity condition on the function $x_k = x_0 + \varepsilon A_k + \varepsilon x_k^*$ in which A_k is an unknown parameter. The expressions for x_k^* and z_k^* are completely determined as a result of substituting B_{k-1}

$$x_k^* = \int_{t_0}^{t}[(f_h')\,z_k^* + (f_{a\varepsilon}'')\,x_{k-1} + (f_{\psi\varepsilon}'')\,y_{k-1} + (f_{h\varepsilon}'')\,z_{k-1} + \frac{1}{2}\,(f_{\varepsilon^2}'') + (f_{ah}'')\,x_{k-1}z_{k-1} + (f_{\psi h}'')\,y_{k-1}z_{k-1} +$$

$$\frac{1}{2}(f_{h^2}'')z_{k-1}^2 + f^*(t_1, x_{k-1}, y_{k-1}, z_{k-1}\varepsilon)\Big]dt_1$$

$$z_k^* = H[(g_{a\varepsilon}'')x_{k-1} + (g_{\psi\varepsilon}'')y_{k-1} + (g_{h\varepsilon}'')z_{k-1} + {}^1/_2(g_{\varepsilon^2}'') + {}^1/_2(g_{h^2}'')z_{k-1}^2 + H'(a^*)x_{k-1}z_{k-1} + g^*(t_1, x_{k-1}, y_{k-1}, z_{k-1}, \varepsilon)]$$

Further, from the periodicity condition of the function $y_k^*(t, A_k, \varepsilon)$

$$y_k^* = \omega'(a^*)A_k(t-t_0) + y_k^{**}(t,\varepsilon) \equiv \omega'(a^*)A_k(t-t_0) + \int_{t_0}^{t}\Big[\omega'(a^*)x_k^* + \frac{1}{2}\omega''(a^*)x_{k-1}^2 + (F_{a\varepsilon}'')x_{k-1} + (F_h')z_k^* +$$
$$(F_{\psi\varepsilon}'')y_{k-1} + (F_{h\varepsilon}'')z_{k-1} + \frac{1}{2}(F_{\varepsilon^2}'') + (F_{ah}'')x_{k-1}z_{k-1} +$$
$$(F_{\psi h}'')y_{k-1}z_{k-1} + \frac{1}{2}(F_{h^2}'')z_{k-1}^2 + F^*(t_1, x_{k-1}, y_{k-1}, z_{k-1}, \varepsilon)\Big]dt_1$$

we can determine A_k, and y_k^*

$$A_k(\varepsilon) = -y_k^{**}(T,\varepsilon)/\omega'(a^*)T, \quad y_k^*(t,\varepsilon) = y_k^{**}(t,\varepsilon) - y_k^{**}(T,\varepsilon)(t-t_0)/T$$

To determine the parameter B_k, we should make use of the expression $x_{k+1}^*(t, B_k, \varepsilon)$

$$x_{k+1}^* = \int_{t_0}^{t} f_{k+1}(t_1,\varepsilon)dt_1 + \int_{t_0}^{t}\{(f_{\psi\varepsilon}'')B_k +$$
$$(f_h')H[(g_{\psi\varepsilon}'')B_k + g^*(t_1, x_k, y_0^* + B_k + \varepsilon y_k^*, z_k, \varepsilon)] +$$
$$(f_{\psi h}'')(z_0 + \varepsilon z_k^*)B_k + f^*(t_1, x_k, y_0^* + B_k + \varepsilon y_k^*, z_k, \varepsilon)\}dt_1$$

$$f_{k+1}(t,\varepsilon) \equiv (f_h')H[(g_{a\varepsilon}'')x_k + (g_{\psi\varepsilon}'')y_k + (g_{h\varepsilon}'')z_k + {}^1/_2(g_{\varepsilon^2}'') + {}^1/_2(g_{h^2}'')z_k^2 + H'(a^*)x_kz_k] + (f_{a\varepsilon}'')x_k + (f_{\psi\varepsilon}'')(y_0^* + \varepsilon y_k^*) +$$
$$(f_{h\varepsilon}'')z_k + {}^1/_2(f_{\varepsilon^2}'') + (f_{ah}'')x_kz_k + (f_{\psi h}'')(y_0^* + \varepsilon y_k^*)z_k + {}^1/_2(f_{h^2}'')z_k^2$$

The periodicity relation for function x_{k+1} is

$$P'(\varphi^*)B_k = -\int_0^T\{f_{k+1}(t,\varepsilon) + f_k^*(t, B_k, \varepsilon) + \varepsilon(f_{\psi h}'')B_kz_k^* + (f_h')H[g_k^*(t, B_k, \varepsilon)]\}dt \tag{2.7}$$

Here the functions f_k^* and g_k^* satisfy a Lipschitz condition in B_k and ε and vanish at $\varepsilon = 0$. Thus, $B_k(0) = B_0$ (see (2.6)). Equation (2.7) complies with all the hypotheses of the existence theorem for the implicit function $B_k(\varepsilon)$ when $\varepsilon > 0$ is sufficiently small. It can be constructed at each step by successive approximations with respect to the small parameter

$$B_k^{(j)}(\varepsilon) = -\frac{1}{P'(\varphi^*)}\int_0^T \{f_{k+1}(t,\varepsilon) + f_k^*(t, B_k^{(j-1)}, \varepsilon) +$$
$$\cdot\, \varepsilon (f''_{\psi h}) B_k^{(j-1)} z_k^* + (f_h') H [g_k^*(t, B_k^{(j-1)}, \varepsilon)]\}\, dt, \quad j = 1, 2, \ldots, k,$$
$$B_k^{(0)} = B_0 \tag{2.8}$$

The successive approximations scheme for constructing the periodic solution x, y, z can be justified on the basis of [4] and we shall not do it here.

The result obtained is expressed by

Theorem 2.1. When $\varepsilon > 0$ is sufficiently small the perturbed system (1.1) admits of a resonance solution of the form (2.1) if

a) conditions (1)-(6) of Sect. 1 are satisfied;

b) a^* and φ^* are simple real roots of the corresponding equations.

Different critical cases are possible when conditions (b) of Theorem 2.1 are not satisfied.

1) Equation (2.4) does not have real roots; then steady-state modes of form (2.1) cannot be realized in system (1.1) no matter how small $\varepsilon > 0$ may be.

2) Equation (2.4) has a multiple real root; in this case an additional investigation, based on Poincaré's method [4], can be carried out for analytic systems. It is well known that as a rule multiple roots lead to a splitting of the trajectories and to expansions in fractional powers of the parameter [5]. A successive approximations scheme has not been developed for this complicated case.

3) Equation (2.4) is satisfied identically, i.e., independently of φ; in this case motions of higher degree can occur [4], for which assertions with remarks (1) and (2), analogous to Theorem 2.1 are established on the basis of Poincaré's method.

4) The frequency $\omega = \mathrm{const}$, where $\omega = n\nu / m$; then the scheme developed above simplifies [4]. The defining equations are

$$P(a,\varphi) \equiv \int_0^T \{(f_\varepsilon') + (f_h') H [(g_\varepsilon')]\}\, dt = 0$$
$$Q(a,\varphi) \equiv \int_0^T \{(F_\varepsilon') + (F_h') H [(g_\varepsilon')]\}\, dt = 0 \tag{2.9}$$

The condition for the existence of a steady-state resonance mode is that the functional determinant $\det J \equiv \det\, (\partial\, (P, Q) \,/\, \partial(a^*, \varphi^*))$ be nonzero. It should be noted that in actual problems the defining equations of type (2.4) or (2.9) can be constructed on the basis of the integrals of the initial system [2].

3. Investigation of Liapunov stability. The stability of the solution constructed cannot be investigated on the basis of the zero-approximation system in ε. To compute the characteristic indices it is necessary to set up the variational system by means of the substitutions

$$a = a(t, \varepsilon) + U, \psi = \psi(t, \varepsilon) + V, h = h(t, \varepsilon) + W$$

It is obvious that when $\varepsilon = 0$ the rest point is unstable for $t \gtreqless t_0$ since one group of solutions corresponds to a two-fold zero characteristic index, i.e., a complicated critical case occurs [2, 4, 6], and the critical characteristic indices are of orders of a fractional power of ε. The Floquet-Liapunov theory [6] is applicable to a variational system with periodic coefficients. On its basis we make the replacement

$$U = ue^{\alpha t}, \ V = ve^{\alpha t}, \ W = we^{\alpha t}$$

where u, v, w are periodic functions of period T and α is the characteristic index. The quantities sought for are determined from the system

$$\begin{aligned} u^{\cdot} &= (f_a' - \alpha)\, u + f_\psi'\, v + f_h' w \\ v^{\cdot} &= (\omega' + F_a')\, u + (F_\psi' - \alpha)\, v + F_h' w \\ w^{\cdot} &= (H'h + g_a')\, u + g_\psi'\, v + (H + g_h' - I\alpha)\, \omega \end{aligned} \tag{3.1}$$

We are required to find the value of α for which system (3.1) admits of a periodic solution. By the successive approximations method [6] we can show that $\alpha = \delta\alpha_1 + \delta^2\alpha_2 + O(\delta^3)$, where $\delta = \sqrt{\varepsilon}$, $u(t, \varepsilon) = u_0 + \delta u_1 + \delta^2 u_2 + \delta^3 u_3\ (t, \varepsilon)$ and analogously for v and w.

From the equations of the zero approximation in δ it follows that $u_0 \equiv 0$, $w_0 \equiv 0$, and $v_0 = \mathrm{const}$. The periodicity condition on functions u_1, v_1, w_1 leads to the expressions

$$u_1 = \mathrm{const}, \quad v_1 = \mathrm{const}, \quad \alpha_1 v_0 = \omega'(a^*) u_1, \ w_1 \equiv 0$$

Substituting $w_2 = v_0 H[(g_{\psi\varepsilon}'')]$ into the equation for u_2 leads to the relation

$$\alpha_1^2 = \omega'(a^*)\, P'(\varphi^*) \,/\, T \tag{3.2}$$

In the approximation computed the critical characteristic indices are pure imaginary when $\alpha_1^2 < 0$. Their computation to within order ε from the periodicity conditions on the zero approximation of the functions u_3, v_3, w_3 leads to the expression

$$\begin{aligned} \alpha_2 = \frac{1}{2T} \int_0^T & [(f_{a\varepsilon}'') + (f_{ah}'')\, z_0 + (f_h')\, \omega'(a^*)\, w_3^* + \\ & (F_{\psi\varepsilon}'') + (F_{\psi h}'')\, z_0 + (F_h')\, w_2^*]\, dt \\ w_2^*(t) = H[(g_{\psi\varepsilon}'')], \quad & w_3^*(t) = H[H'(a^*) z_0 \,/\, \omega'(a^*) + \\ & (g_{a\varepsilon}'') \,/\, \omega'(a^*) - w_2^*] \end{aligned} \tag{3.3}$$

Theorem 3.1. The perturbed solution (2.1) of system (1.1) is asymptotically stable when $\varepsilon > 0$ is sufficiently small, if

a) $\alpha_1^2 < 0$ is the necessary condition (see (3.2));

b) $\alpha_2 < 0$ (see (3.3))

and is unstable if even one reverse inequality holds.

For the case $\omega = \mathrm{const}$, considered in (4) in Sect. 2, the sufficient conditions for asymptotic stability are the requirements that the real parts of both roots of the quadratic equation $\det(J - I\alpha) = 0$ be negative.

In the special case of system (1.1), when vector h is absent (the perturbing terms are proportional to ε) the quantities $P(\varphi^*)$ and α_2 have a simpler form [7]

$$P(\varphi) = \int_0^T (f)\, dt, \quad \alpha_2 = \int_0^T [(f_a') + (F_\psi')]\, dt \quad (f \to \varepsilon f, \ \ F \to \varepsilon F)$$

In many actual problems the stability conditions can be written out from the integrals of the initial unperturbed system. For example, for the nearly-conservative rotary system [8]: $x^{\cdot\cdot} + Q(x) = \varepsilon q(\nu t, x, x^\cdot, \varepsilon)$, the defining relation (2.4) and the stability conditions are

$$P(\varphi) \equiv \int_0^T q(\nu t, x_0, x_0^\cdot, 0)\, x_0^\cdot dt = \int_0^{2\pi} q\left(\nu \int \frac{dx}{x_0^\cdot(x, a^*)} - \right.$$

$$\left. \nu\varphi, x.\, x_0^\cdot(x, a^*), 0\right) dx = 0, \qquad x_0^\cdot = [2(a^* - \int Q dx)]^{1/2}$$

$$\text{a)} \ \ P'(\varphi^*) < 0, \quad b) \int_0^{2\pi} (q_{x^\cdot}') \frac{dx}{x_0^\cdot} < 0$$

In the case $\alpha_2 = 0$ it becomes necessary to compute the critical characteristic indices more exactly in the powers of δ, and that calls for greater smoothness of system (1.1).

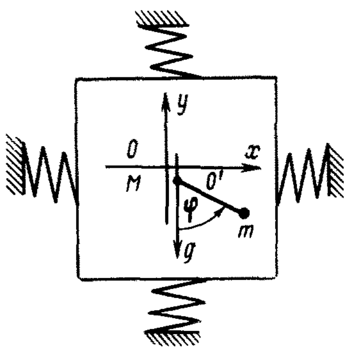

Fig. 1

4. Example. We examine the problem of the steady-state oscillations and rotations in the plane of a system with three degrees of freedom, shown in Fig. 1. Here M is the mass of the foundation in which an unbalanced rotor is located; m is the mass rotor, l is the arm, φ is the angular deviation. The origin is located at point O at which the rotor spin axis O' is located when the system is in balance. We assume that the foundation can accomplish only translational oscillatory motions with respect to the x- and y-axes. Then, the model's equations of motion, with due regard to the periodic external forces and to viscous friction, can be written as

$$(M + m) X^{\cdot\cdot} + ml(\varphi^\cdot \cos\varphi)^\cdot + K_x X + Q_x(X, Y) = -\mathrm{B}_x X^\cdot + F_x(\nu t)$$

$$(M + m) Y^{\cdot\cdot} + ml(\varphi^\cdot \sin\varphi)^\cdot + K_y Y + Q_y(X, Y) = -\mathrm{B}_y Y^\cdot + F_y(\nu t) \tag{4.1}$$

$$ml^2\varphi^{\cdot\cdot} + ml(X^\cdot \cos\varphi + Y^\cdot \sin\varphi)^\cdot + mgl\sin\varphi = -\mathrm{B}l\varphi^\cdot - \Gamma l(X^\cdot\cos\varphi + Y^\cdot\sin\varphi) + N(\nu t);$$

Here X and Y are the Cartesian coordinates of point O'; K_x and K_y are the coefficients of elasticity; Q_x and Q_y are nonlinear components whose expansions begin with quadratic terms; B_x and B_y are coefficients of viscous friction; F_x and F_y are periodic external forces; ν is the frequency. Viscous frictional forces with coefficients B and Γ, the periodic external moment N and the gravitational moment (g is the free-fall acceleration) act on the rotor. We assume that all the system parameters are constants.

Below we examine the case, being of practical interest, of a spinning rotor. We assume that the frequency ν of the external force, of the order of the natural frequencies of the oscillations of the foundation, fastened viscoelastically, is much greater that the characteristic frequency of oscillations of the rotor in the gravitational force field $\sqrt{g/l}$, i.e., $g/l\nu^2 \sim \varepsilon$. Then under certain natural assumptions the rotor's steady-state motion is close to an uniform rotation with velocity $\omega \sim \nu$, while the oscillations of the foundation relative to the equilibrium position take place with a frequency $\sim \nu$ and an amplitude of the order of ε. With due regard to the assumptions mentioned, whose physical sense is clear, we further introduce dimensionless variables and parameters of the system

$$\frac{m}{M+m} = \varepsilon \ll 1, \quad s = \nu t, \quad \frac{g}{l\nu^2} = \varepsilon\omega_0^2 \quad (\omega_0 \sim 1)$$
$$\frac{X}{l} = x, \quad \frac{Y}{l} = y, \quad \frac{Q_{x,y}(X, Y)}{(M+m)l\nu^2} = q_{x,y}(x, y)$$

$$\frac{K_{x,y}}{(M+m)\nu^2} = \omega_{x,y}^2, \quad \frac{B_{x,y}}{(M+m)\nu} = \beta_{x,y}, \quad \frac{F_{x,y}(\nu t)}{(M+m)l\nu^2} = \varepsilon f_{x,y}(s)$$
$$\frac{B}{ml\nu} = \varepsilon\beta, \quad \frac{\Gamma}{ml\nu} = \varepsilon\gamma, \quad \frac{N(\nu t)}{ml^2\nu^2} = \varepsilon\mu(s)$$

A division of the first two equations in (4.1) by $(M+m)\,l\nu^2$, of the third equation by $ml^2\nu^2$, and the use of the notation adopted after the system has been reduced to a normal form (solvable relative to the leading (second) derivatives with respect to the "fast dimensionless time" s) enables us right away to write it in the form of system (1.1). The variables x and y and their derivatives with respect to s: $x' = u$ and $y' = v$ form a stable fourth-order vector h, while as the phase ψ we can take the angular variable φ, whose rotation velocity $\varphi' = \omega$ is the quasi-static variable a in the notation of Sect. 1. As a result we obtain a system of six equations of form (1.1)

$$\begin{aligned}
&\omega' = f, \quad \varphi' = \omega \qquad (' \equiv d/ds)\\
&x' = u, \quad u' = -\beta_x u - \omega_x^2 x - q_x + \varepsilon(f_x + \omega^2 \sin\varphi - f\cos\varphi)\\
&y' = v, \quad v' = -\beta_y v - \omega_y^2 y - q_y + \varepsilon(f_y - \omega^2\cos\varphi - f\sin\varphi) \qquad (4.2)\\
&f \equiv f(s, \omega, \varphi, x, u, y, v, \varepsilon) =\\
&\quad \cos\varphi\,(1-\varepsilon)^{-1}(\beta_x u + \omega_x^2 x + q_x - \varepsilon f_x) + \sin\varphi\,(1-\varepsilon)^{-1}(\beta_y v +\\
&\quad \omega_y^2 y + q_y - \varepsilon f_y) + \varepsilon\,(1-\varepsilon)^{-1}[\mu - \omega_0^2\sin\varphi - \beta\omega - \gamma(u\cos\varphi +\\
&\qquad v\sin\varphi)]
\end{aligned}$$

(the function $F \equiv 0$ and the matrix H is constant).

Let $f_x = f_{x0} \sin (s + \delta_x)$ and $f_y = f_{y0} \sin (s + \delta_y)$; then in the first approximation the stable elements x and y are sums of two harmonic functions of s with frequencies 1 and n / m of the form (τ is the phase constant)

$$x = \varepsilon A_x \sin (s + \alpha_x) + \varepsilon B_x \sin \left(\frac{n}{m} s + \tau + \tau_x \right)$$

$$A_x = f_{x0} \left[(\omega_x^2 - 1)^2 + \beta_x^2\right]^{-1/2}, \quad B_x = \left(\frac{n}{m}\right)^2 \left[\left(\omega_x^2 - \frac{n^2}{m^2}\right)^2 + \beta_x^2 \frac{n^2}{m^2}\right]^{-1/2}$$

$$\alpha_x = \operatorname{arctg} \left[\beta_x (1 - \omega_x^2)^{-1}\right] + \delta_x, \quad \tau_x = \operatorname{arctg} \left[\beta_x \frac{n}{m} \left(\frac{n^2}{m^2} - \omega_x^2\right)^{-1}\right]$$

The expression for y is obtained by replacing the subscript x by y. The substitution of functions x and y and of their derivatives u and v into the first equation of (4.2) leads to an equation of type (2.4) for the determination of the phase constant, from which it follows that a fundamental resonance mode can be realized in the system, i.e., $n = m = 1$

$$\begin{aligned} &2P(\tau) \equiv r \sin (\tau + \theta) + c = 0, \quad \tau^* = \tau_{1,2} = -\operatorname{Arcsin} c/r - \theta \\ &r^2 \equiv a^2 + b^2 = (\beta_x A_x \cos \alpha_x + \omega_x^2 A_x \sin \alpha_x - f_{x0} \cos \delta_x - \beta_y A_y \sin\alpha_y + \\ &\quad \omega_y^2 A_y \cos \alpha_y - f_{y0} \cos \delta_y)^2 + (\beta_x A_x \sin \alpha_x - \omega_x^2 A_x \cos \alpha_x + f_{x0} \cos \delta_x + \\ &\quad \beta_y A_y \cos \alpha_y + \omega_y^2 A_y \sin \alpha_y - f_{y0} \sin \delta_y)^2 \\ &c = (\beta_x \cos \tau_x + \omega_x^2 \sin \tau_x) B_x + (\beta_y \sin \tau_y - \omega_y^2 \cos \tau_y) B_y + \mu_0 - \beta \\ &\operatorname{tg} \theta = \frac{a}{b}, \quad \mu_0 = \frac{1}{2\pi} \int_0^{2\pi} \mu(s)\, ds \end{aligned} \tag{4.3}$$

Equation (4.3) admits of two simple real roots $\tau^* = \tau_{1,2}$ in the interval $\tau \in [0, 2\pi]$ if $|c / r| < 1$, which we assume. Then $P'(\tau^*) \neq 0$ and all the conditions in Theorem 2.1 on the existence and uniqueness of a steady-state resonance solution corresponding to each root $\tau_{1,2}$ are satisfied. This solution can be constructed to any degree of accuracy by a series expansion or by successive approximations with respect to ε using the procedure in Sect. 2.

From the form of function $P(\tau)$, in (4.3) it follows that for one of the roots, say $\tau^* = \tau_1$, $P'(\tau_1) < 0$, while for the other, $P'(\tau_2) > 0$. On the basis of Theorem 3.1 the steady-state mode corresponding to root τ_2 is unstable. The motion corresponding to root τ_1 is stable and, moreover, asymptotically stable when $\varepsilon > 0$ is sufficiently small, if $\beta > {}^1/_2\delta$

$$\begin{aligned} \delta = {} & r_x^{-1} \left\{\beta_x a_x + [\omega_x^2 (\omega_x^2 - 1) + \beta_x^2]\, b_x + \beta_x (\omega_x^2 - 1)\right\} + \\ & r_y^{-1} \left\{[\omega_y^2 (\omega_y^2 - 1) + \beta_y^2]\, a_y + \beta_y b_y + \beta_y (\omega_y^2 - 1)\right\} \end{aligned}$$

$$a_x = 2 + r_x^{-1} [2 (\omega_y^2 - 1) - \beta_x^2], \quad b_x = -\beta_x r_x^{-1} (\omega_x^2 + 1), \quad r_x = (\omega_x^2 - 1)^2 + \beta_x^2$$

$$a_y = -\beta_y r_y^{-1} (\omega_y^2 + 1), \quad b_y = -2 - r_y^{-1} [2 (\omega_y^2 - 1) - \beta_y^2], \quad r_y = (\omega_y^2 - 1)^2 + \beta_y^2$$

It should be noted that the steady-state mode can take place if at least one of the quantities f_{x0} or f_{y0} is nonzero. Otherwise, spinning oscillatory motions cannot be realized in the system when $\mu_0 \neq \beta$. However, if $\mu_0 = \beta$ the system (4.2) can admit of steady-state resonance motions of higher degrees [4]. This case requres additional investigation since $P(\tau) \equiv 0$ (see (3) in Sect. 2).

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